

FRACAL STRINGS AS THE BASIS OF CANTORIAN-FRACTAL SPACETIME AND THE FINE STRUCTURE CONSTANT

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Abstract

Beginning with the most general fractal strings/sprays construction recently expounded in the book by Lapidus and Frankenhuisen, it is shown how the *complexified* extension of El Naschie's Cantorian-Fractal spacetime model belongs to a very special class of families of fractal strings/sprays whose scaling ratios are given by suitable pinary (pinary, p prime) powers of the Golden Mean. We then proceed to show why the logarithmic periodicity laws in Nature are direct physical consequences of the complex dimensions associated with these fractal strings/sprays. We proceed with a discussion on quasi-crystals with p -adic internal symmetries, von Neumann's Continuous Geometry, the role of wild topology in fractal strings/sprays, the Banach-Tarski paradox, tessellations of the hyperbolic plane, quark confinement and the Mersenne-prime hierarchy of bit-string physics in determining the fundamental physical constants in Nature.

1. Introduction: Fractal strings

We will briefly summarize the basic ideas behind the book by [1] on Fractal strings. A standard fractal string \mathcal{L} is a bounded open subset Ω of the real line R . It is well known to the experts that such a set consists of countably many open intervals, the lengths of which will be denoted by $l_1, l_2, l_3, \dots, l_j$. These are called the lengths of the string. The sum $\sum l_j$ is finite and equals the Lebesgue measure of Ω . Despite the fact that one is dealing with countably many intervals/lengths in the definition of Ω the *boundary* $\partial\Omega$ is not necessarily countably finite. The *boundary* of the Cantor string is the ternary Cantor set which is a *non* countable dust of points (it has the same cardinality as the real line) despite having zero measure.

The geometric counting function of the lengths is defined as:

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} (l_j)^s. \quad (1)$$

The *central* notion of the book [1] is that this function of s has poles at complex numbers s_n , the so-called complex dimensions of a fractal string. The spectrum of a fractal string consists of the sequence of frequencies : $f = k/l_j$ for $k, j = 1, 2, 3, \dots$. The spectral zeta function of the fractal string \mathcal{L} is defined as:

$$\zeta_{\nu}(s) = \sum_f f^{-s}. \quad (2)$$

The geometry and the spectrum of \mathcal{L} are connected by the following relationship, valid for all values of s :

$$\zeta_{\nu}(s) = \zeta_{\mathcal{L}}(s)\zeta_R(s). \quad (3)$$

where the Riemann zeta function is:

$$\zeta_R(s) = \sum n^{-s}. \quad (4)$$

it has a simple pole at $s = 1$ with residue 1 and it admits an analytic continuation to the whole complex plane. The Cantor string is defined by constructing the sequence of lengths using two scaling factors $r_1 = r_2 = 1/3$. One starts with the unit interval and scales it by these two scaling factors yielding two segments of length equal to $1/3$. One then repeats this process iteratively yielding the Cantor string which is a self similar string

consisting of segments of lengths 3^{-n} with multiplicities of $1, 2, 4, \dots, 2^n$ respectively. The geometric counting function of the Cantor string is then given by the geometric series:

$$\zeta_{CS}(s) = \sum 2^{-n} 3^{-ns} = \frac{1}{1 - 23^{-s}}. \quad (5)$$

The complex dimensions are given by the poles of this function, zeros of the denominator:

$$23^{-s} = 1 = 1e^{i2\pi n}. \quad (6)$$

Taking logarithms on both sides of this equation yields:

$$\ln 2 - s \ln 3 = 0 \pm i2\pi n \Rightarrow s = \frac{\ln 2}{\ln 3} \pm \frac{i2\pi n}{\ln 3}. \quad (7)$$

we immediately can recognize that the real part of the complex dimension $\ln 2 / \ln 3$ coincides precisely with the fractal dimension of the ternary Cantor set. The *boundary* of the fractal string \mathcal{L} is precisely the uncountable Cantor fractal dust of points. There are many types of fractal strings discussed in the book [1]. In particular, the Fibonacci fractal string which is constructed using the scaling ratios $r_1 = 1/2$ and $r_2 = 1/4$. The sequence of lengths are $1, 1/2, 1/4, 1/8, \dots, 1/2^n, \dots$ with respective multiplicities given by the Fibonacci numbers $1, 1, 2, 3, \dots, F_{n+1}, \dots$ which obey $F_{n+1} = F_n + F_{n-1}$ and whose ratio in the large n limit is given by the Golden ratio:

$$\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \tau = 1 + \phi = \frac{1 + \sqrt{5}}{2} = 1.618... \quad (8)$$

The geometric counting function of the Fibonacci string is:

$$\zeta_{Fib}(s) = \frac{1}{1 - 2^{-s} - 4^{-s}}. \quad (9)$$

and the poles are found by solving $(2^{-s})^2 + 2^{-s} = 1$ which gives the complex dimensions:

$$\begin{aligned} & \frac{\ln \tau}{\ln 2} \pm \frac{i2\pi n}{\ln 2} \\ & - \frac{\ln \tau}{\ln 2} \pm \frac{i2\pi(n + 1/2)}{\ln 2}. \end{aligned} \quad (10)$$

Notice that *negative* dimensions are very natural ingredients of the complex dimensions of fractal strings. Negative dimensions fit the negative-entropy proposal of Michael Conrad [3] to understand living systems. Both the Cantor and Fibonacci strings belong to the class of lattice strings, meaning that they have oscillations in the geometry of the order of $D = \text{Real}(s)$. We can hear the dimension of fractal strings [1]. We can also hear if a string is Minkowski measurable if, and only if, the $\zeta(s)$ has *no* zeros on the line $\text{Real}(s) = D$. Cantor strings, lattice strings, are not Minkowski measurable. A class of nonlattice strings is given, for example, for scaling ratios given by $r_1 = 1/2$ and by $r_2 = 1/3$; i.e the scaling ratios are not given by powers of a fixed $r < 1$. Another example is given by the so-called Golden string in [1] whose scaling ratios are $r_1 = 1/2$ and $r_2 = (1/2)^{1+\phi}$. In particular, the complex dimensions of these strings are given by solving a *transcendental* equation which yields a *nonlattice* structure of points in the complex planes. For this reason these strings are called non-lattice, despite being self-similar. The complex dimensions of the Golden string is given by an almost periodic structure in the complex plane. All complex dimensions of a self similar string with scaling ratios r_1, r_2, \dots, r_N lie to the left of, or on the line $\text{Real}(s) = D$ [1]. In order not to confuse the reader with the nomenclature used in [1] we emphasize that our construction of Cantorian-Fractal spacetime [2] is based entirely in a particular class of fractal strings, and higher dimensional fractal sprays or branes, based on suitable binary powers and pinary powers of the Golden mean. By pinary we mean powers of a prime number p :

$$2^{-\phi^j} \cdot p^{-\phi^j} \cdot j = \pm 1, \pm 2, \pm 3, \dots \quad (11)$$

In section 3 we will show that complex dimension is not just a mathematical artifact but that is deeply related to the log-periodic laws in Nature discussed amply by Nottale et al [4,5] in their theories of the Fractal Tree of life and Fractal Evolution and by many others in particle physics in the normalization group context [6,7]. We believe that quantum gravitational phenomena should involve *interference* of complex dimensions. With this preamble of some of the basic ideas on fractal strings of the book [1] we turn attention to the construction of El Naschie's Cantorian-Fractal spacetime and its complexification. Before doing so, we must discuss the notion of fractal sprays (branes) which are the higher dimensional analogs of strings.

2. Fractal Branes/Sprays

In this section we are going to generalize the construction of fractal strings to the p -branes case. In particular we will be studying a subclass of fractal branes called *sprays* in the literature [1]. Having done so we will show how the Cantorian-Fractal spacetime model [2] is a very special representative of these fractal sprays models given by [1] and which are based on scaling ratios given by powers of the golden mean ϕ^j , where $j = \pm 1, \pm 2, \pm 3, \dots$. A self-similar fractal spray [1] Ω with basic shape B , scaled by a self-similar string \mathcal{L} , whose dimension can be *greater than one*, is given by a collection of sets Ω_j which are congruent to $l_j B$, the homothetic of Ω by the ratio $l_j < 1$ for each value of j . Hence, for instance, the fractal spray of the Cantor string on the unit square B , see figure, is obtained by having one open unit square; two open squares of length-size $1/3$, four open squares of length-size $1/9$...and so forth. The spectral zeta function of the Dirichlet Laplacian on the square is:

$$\zeta_B(s) = \sum_{n_1, n_2} (n_1^2 + n_2^2)^{-s/2}. \quad (12)$$

and the spectral zeta function of the spray associated with the Cantor-string is defined by:

$$\zeta_\nu(s) = \zeta_{CS}(s)\zeta_B(s) \quad (13)$$

where the geometric length counting function of the Cantor string was given in the previous section:

$$\zeta_{CS}(s) = \sum_j (l_j)^s = \sum_j 2^n 3^{-ns} = \frac{1}{1 - 23^{-s}}. \quad (14)$$

The $\zeta_B(s)$ has poles at $s = 2$ and the $\zeta_{CS}(s)$ has poles at: $s = (\ln 2 / \ln 3) \pm i2\pi n / \ln 3$. The Sierpinski drum is also a fractal spray associated with the unit area triangle \mathcal{T} and is obtained by scaling the middle triangle with the scaling ratios: $r_1 = r_2 = r_3 = 1/2$. Notice now that $\sum r_j = 3/2 > 1$ and for this reason the dimension of this fractal spray will be greater than unity. There is one triangle of unit area, 3 triangles of $1/4$ area, 9 triangles of $1/16$ area, and so forth. The geometric *length* counting function associated with the scaling lengths $r_1 = r_2 = r_3 = 1/2$ is:

$$\zeta_{\mathcal{L}}(s) = \sum (l_j)^s = \sum 3^{-n} 2^{-ns} = \frac{1}{1 - 32^{-s}}. \quad (15)$$

and has poles at:

$$s = \frac{\ln 3}{\ln 2} \pm \frac{i2\pi n}{\ln 2}. \quad (16)$$

Notice that the real part of s is given by $D = \ln 3 / \ln 2 > 1$ which is precisely the fractal dimension of the Sierpinski's gasket. The spectral zeta function of the Dirichlet Laplacian on the unit triangle is:

$$\zeta_{\mathcal{T}}(s) = \sum_{m,n} (m^2 + mn + n^2)^{-s/2}. \quad (17)$$

and it has poles at $s = 2$ with residue $(\pi/3\sqrt{3})$ and at $s = 1$ with residue $-3/4$. The spectral zeta of the Sierpinski drum is finally given by:

$$\zeta_\nu(s) = \zeta_{\mathcal{T}}(s) \frac{1}{1 - 32^{-s}}. \quad (18)$$

Notice that *no* complex dimension of \mathcal{L} (the string whose scaling ratios $r_1 = r_2 = r_3 = 1/2$ were used to generate the Sierpinski drum whose real dimension is given by $D = \ln 3 / \ln 2$) *coincides* with the poles of $\zeta_{\mathcal{T}}(s)$, given by $s = 2$ and $s = 1$. For this reason D is also the dimension of the *boundary* of the fractal spray Ω associated with the string \mathcal{L} . Hence we have that:

$$d - 1 \leq D = \dim \partial\Omega \leq d \Rightarrow 1 \leq \frac{\ln 3}{\ln 2} \leq 2. \quad (19)$$

where $d = 2$ is the dimension of the ambient embedding space R^2 used in the construction of the fractal spray Ω .

3. Cantorian-Fractal spacetime as a very special class of Fractal strings/sprays

Having outlined how to construct a fractal spray in the ambient space R^2 will allow us to show why Cantorian-Fractal spacetime

$\mathcal{E}^{(\infty)}$ [2] comprised of an infinite hierarchy of sets $\mathcal{E}^{(j)}$ of dimension:

$$(1 + \phi)^{j-1}. \quad j = 0, \pm 1, \pm 2, \pm 3, \dots, \pm \infty. \quad (20)$$

is a special class of fractal strings/sprays [1] whose scaling ratios are suitable binary powers of the Golden mean $2^{-\phi^{j-1}}$. We shall call these fractal sprays the Golden strings/sprays and must *not* be confused with the Golden strings discussed by the authors [1]. The latter are *nonlattice* self similar strings whereas the former are lattice self similar ones. Let us consider the Golden fractal spray Ω obtained by scaling an open square B of unit area by the scaling lengths:

$$r_1 = r_2 = 2^{-\phi}. \quad (21)$$

Thus Ω is a bounded open subset of R^2 consisting of one open square of unit area, 2 open squares of length-size $2^{-\phi}$ (area is $2^{-2\phi}$), 4 open squares of length size $2^{-2\phi}$ (area is $2^{-4\phi}$), etc.... The geometric length counting function associated with the scaling: $r_1 = r_2 = 2^{-\phi}$ is:

$$\zeta_{\mathcal{L}}(s) = \sum (l_j)^s = \sum 2^n 2^{-n\phi s} = \frac{1}{1 - 22^{-\phi s}}. \quad (22)$$

and it has poles when the denominator vanishes:

$$1 = 1e^{i2\pi n} = 22^{-\phi s} \Rightarrow 0 \pm i2\pi n = \ln 2 - \phi s \ln 2 \Rightarrow$$

$$s = (1 + \phi) \pm \frac{i2\pi n(1 + \phi)}{\ln 2} \quad (23)$$

where we have used the defining relation of the Golden Mean:

$$\frac{1}{\phi} = 1 + \phi \Rightarrow 1 = \phi + \phi^2 \Rightarrow \phi = \frac{\sqrt{5} - 1}{2} = 0.618... \quad (24)$$

It is not difficult to prove that the Golden mean generates a ring structure $Z[\tau]$, where $\tau = 1 + \phi$:

$$(1 + \phi)^n = F_{n+1} + \phi F_n. \quad \phi^n = (-1)^n F_{n-1} + (-1)^{n+1} F_n \phi. \quad (25)$$

where F_n are the Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, 21... obeying the recursive relation: $F_{n+1} = F_n + F_{n-1}$ and the limit of $F_{n+1}/F_n = 1 + \phi$ when $n \rightarrow \infty$. Ring structures are essential ingredients in von Neumann's formulation of Continuous Geometry [8]. The spectral Dirichlet Laplacian of the unit square is:

$$\zeta_B(s) = \sum (n_1^2 + n_2^2)^{-s/2} \quad (26)$$

and it has poles at $s = 2$. Hence the spectral zeta function of this fractal Golden spray is:

$$\zeta_\nu(s) = \frac{1}{1 - 22^{-\phi s}} \zeta_B(s). \quad (27)$$

Since the poles of $\zeta_B(s)$ do *not* coincide with the zeros of the denominator $1 - 22^{-\phi s} = 0$ we have that the dimension of the Golden spray is precisely equal to the dimension of the *boundary* of the open bounded domain Ω : $\partial\Omega$. Hence we have that:

$$d - 1 \leq (1 + \phi) \equiv \dim \partial\Omega \leq d \Rightarrow 1 \leq 1 + \phi \leq 2. \quad (28)$$

This procedure can be generalize to higher-dimensions and in this fashion we will be able to construct a physical model of Cantorian-Fractal spacetime [2] from the fractal sprays associated with the basic domains given by the unit size hypercubes of increasing dimensionality. Since $1 + \phi < 2$ we were able to construct the set $\mathcal{E}^{(2)}$ based on scaling of two-dim square domains. Because $(1 + \phi)^2 = 2 + \phi < 3$ we can construct the set $\mathcal{E}^{(3)}$ by a similar procedure of scaling the 3-dim unit cube by factors of $r_1 = r_2 = 2^{-\phi^2}$. The geometric length counting function in this case is:

$$\zeta_{\mathcal{L}}(s) = \sum 2^n 2^{-n\phi^2 s} = \frac{1}{1 - 22^{-\phi^2 s}}. \quad (29)$$

and has poles at the zeros of the denominator:

$$1 = 1e^{i2\pi n} = 22^{-\phi^2 s} \Rightarrow 0 \pm i2\pi n = \ln 2 - \phi^2 s \ln 2 \Rightarrow$$

$$s = (1 + \phi)^2 \pm \frac{i2\pi n(1 + \phi)^2}{\ln 2}. \quad (30)$$

Hence we can see that the real part of the complex dimension coincides with the corresponding dimension of the set:

$$(1 + \phi)^2 = 2 + \phi = \dim \mathcal{E}^{(3)}$$

The spectral zeta function of the Dirichlet Laplacian on the unit cube is :

$$\zeta_B(s) = \sum (n_1^2 + n_2^2 + n_3^2)^{-s/2}. \quad (31)$$

and it has poles at $s = 3$. The dimension of this Golden spray coincides with the dimension of the *boundary* of the 3-dimensional open domain Ω living in the ambient space R^3 :

$$d - 1 \leq (1 + \phi)^2 \equiv \dim \partial\Omega \leq d \Rightarrow 2 \leq (1 + \phi)^2 = 2 + \phi \leq 3. \quad (34)$$

This construction generalizes to the full space $\mathcal{E}^{(\infty)}$ with the provision that one constructs the Golden sprays from suitable hypercubes of unit hypervolumes and of *enough* dimensionality N to obey:

$$N > (1 + \phi)^n = F_{n+1} + \phi F_n > N - 1. \quad (35)$$

For example, to construct the set $\mathcal{E}^{(4)}$ whose dimension is $(1 + \phi)^3 = 4 + \phi^3 = 3 + 2\phi > 4$ requires starting with a unit size hypercube in R^5 , instead of R^4 !. To construct the set $\mathcal{E}^{(5)}$ whose dimension is $(1 + \phi)^4 = 5 + 3\phi > 6$ requires starting with a unit size hypercube in R^7 . And in general to construct the set $\mathcal{E}^{(j)}$ whose dimension is $(1 + \phi)^{j-1}$ requires to find the smallest integers $N(j)$ such that:

$$N(j) > (1 + \phi)^{j-1} = F_j + \phi F_{j-1} > N(j) - 1. \quad (36)$$

and to begin the construction of the Golden spray by starting with a unit size hypercube in $R^{N(j)}$ whose scaling ratios are:

$$r_1 = r_2 = 2^{-\phi^{j-1}}. \quad j = 2, 3, 4, 5, \dots \quad (37)$$

The poles of the corresponding geometric counting length:

$$\zeta_{\mathcal{L}}(s) = \sum 2^n 2^{-n\phi^{j-1}s} = \frac{1}{1 - 22^{-\phi^{j-1}s}}. \quad (38)$$

are obtained as usual, from the zeros of the denominator:

$$1 = 1e^{i2\pi n} = 22^{-\phi^{j-1}s} \Rightarrow 0 \pm i2\pi n = \ln 2 - \phi^{j-1}s \ln 2 \Rightarrow$$

$$s = (1 + \phi)^{j-1} \pm \frac{i2\pi n(1 + \phi)^{j-1}}{\ln 2}. \quad (39)$$

Hence, the dimension of the fractal *boundary* of the domain Ω living in the ambient space $R^{N(j)}$ coincides precisely with the dimension of the set $\mathcal{E}^{(j)}$:

$$N(j) - 1 \leq (1 + \phi)^{j-1} = \dim \partial\Omega \leq N(j). \quad (40)$$

This procedure also works for sets whose dimensionality is less than one :

$$0 \leq D \leq 1. \quad (41)$$

By starting with the interval $(0, 1)$ one may construct a hierarchy of fractal strings of dimensionality:

$$0 \leq \phi^{j-1} \leq 1. \quad (42)$$

by using the scaling ratios $r_1 = r_2 = 2^{-(1+\phi)^{j-1}}$. The corresponding geometric counting lengths of this hierarchy of fractal strings are:

$$\zeta_{\mathcal{L}} = \sum 2^n 2^{-n(1+\phi)^{j-1}s} = \frac{1}{1 - 22^{-(1+\phi)^{j-1}s}}. \quad (43)$$

whose poles are located at:

$$s = \phi^{j-1} \pm \frac{i2\pi n\phi^{j-1}}{\ln 2}. \quad (44)$$

The dimensions of this hierarchy of fractal strings is $0 \leq D \leq 1$ agree with the values of the dimension function of von Neumann's Continuous Geometry [8]. Notice that the normal set $\mathcal{E}^{(1)}$ has dimension equal to $D = (1 + \phi)^{1-1} = 1$. This set corresponds to the interval $(0, 1)$. The scaling ratios in this limiting case are $r_1 = r_2 = 1/2$ which entails that if we scale the unit interval by $r_1 = r_2 = 1/2$ and join-in the *two* segments of length $1/2$ we will get back the unit interval. Repeating this procedure ad infinitum one we will always end back with the unit interval. This normal set is not fractal. Concluding, we have shown how Cantorian-Fractal spacetime belongs to a very special class of fractal strings/sprays whose scaling ratios are given by suitable binary powers of the Golden Mean. This procedure automatically furnishes the topology of $\mathcal{E}^{(\infty)}$ displayed by an infinite collection of sets as show in figure, for the set $\mathcal{E}^{(2)}$. We could have built the fractals sprays differently, by scaling given by :

$$p^{-\phi^{j-1}}. \quad p = \text{prime}. \quad (45)$$

. For example, taking $p = 3$ our basic domain would have been a triangle. Then we scale this triangle by the scaling functions $r_1 = r_2 = r_3 = 3^{-\phi}$ to generate 3 congruent triangles of areas $3^{-2\phi}$ smaller than the original and that are attached symmetrically to the 3 sides of the unit triangle. Repeating this process ad infinitum will yield $9, 27, \dots 3^n \dots$ scaled-down versions of the unit triangle and allows to construct a model of the set $\mathcal{E}^{(2)}$ as well. The geometric counting function will be:

$$\zeta_{\mathcal{T}}(s) = \sum p^n p^{-n\phi^{j-1}s} = \frac{1}{1 - pp^{-\phi^{j-1}s}}. \quad (46)$$

whose poles are at:

$$s = (1 + \phi)^{j-1} \pm \frac{i2\pi n(1 + \phi)^{j-1}}{\ln p}. \quad (47)$$

The real part of the dimension coincides again with the dimension of the set $\mathcal{E}^{(j)} = (1 + \phi)^{j-1}$. Only the periods of the imaginary components will vary accordingly to different values of $p = 2, 3, 5, 7, \dots$. It is essential to use scaling given by powers of primes: *pinary* powers. These construction of these fractal sprays by scaling of the unit size domain and their subsequent iterated attachings resembles nothing but a *crystal* growth process; i.e and assembly of hypersurfaces. This assembly of surfaces occurs in the theory of *capped gropes* and link homotopy associated with four-manifolds and belongs to a branch of mathematics called Geometric Topology. The number of surfaces that are attached grow like $2^h - 1$, where h is the grope height. One could envision these fractal sprays as nontrivial embedding of circles and (higher-dimensional) spheres into the ambient spaces R^n . The most famous examples of nontrivial embedding are the Antoine's necklace (a wild knot) and Alexander's Horned Spheres. Their exteriors in R^3 are *not* simply connected and hence are not homeomorphic to the standard embedding of a circle and a sphere. For this reason, these objects belong to what is called wild topology. For recent work on this topic see [9].

In the last section we will discuss further the p -adic Topology and the p -adic internal symmetries of quasi-crystals and their relation to the Fibonacci modules, the Dirichlet zeta, function, the ring of $Z[\tau]$ [10]. It is this subject that will bring us to bit-strings physics, the Mersenne prime hierarchy, Set Theory, the Banach-Tarski paradox, tessellations of hyperbolic planes...and other topics in relation to quark confinement and the values of the fundamental constants in Nature.

4. Log-periodicity as the physical basis of Complex Dimensions

In the past years a lot of activity has been concentrated on the log-periodic laws associated with the fractal structures of evolutionary trees. In particular, the time sequences of major evolutionary leaps at various time scales [5]. Models of this type have been observed in economical crisis patterns in Western pre-Columbian civilizations. The physical model underlying the appearance of such laws is that of critical phenomena [6,7]. The Renormalization Group approach predicts both power laws and logarithmic-periodic corrections.

Nottale et al [5] have considered the simplest Galilean-like renormalization group like equations for the variations of a non-differentiable fractal function $L(x, \epsilon)$ than depends on x and the resolutions ϵ with respect to logarithmic scaling:

$$\frac{\partial L(x, \epsilon)}{\partial \ln \epsilon} = \beta(L). \quad (46)$$

Assuming a "beta" function proportional to L :

$$\beta(L) = \delta L \Rightarrow L(x, \epsilon) = L_o(x) \left(\frac{\epsilon_o}{\epsilon}\right)^\delta. \quad (48)$$

where $\epsilon_o = \lambda$ is the transition "scale" from a fractal to non-fractal behavior of the physical system under consideration. The scaling exponent $\delta = D - D_T$, where D_T is the topological dimension and D is the fractal dimension. A space-filling curve (a Peano curve) has $D_T = 1$ but $D = 2 > D_T$. If the fractal dimension D is complex valued, $d_x + id_y$, then one can see that the imaginary component will be responsible for an oscillatory behavior which yields the logarithmic-periodic character to the function:

$$\begin{aligned} L &= L_o(x) \left(\frac{\epsilon_o}{\epsilon}\right)^{id_y} = L_o(x) \exp[id_y \ln(\epsilon_o/\epsilon)] \Rightarrow \\ L &= L_o(x) \cos[d_y \ln(\frac{\epsilon_o}{\epsilon}) + \alpha]. \quad \alpha = \text{phase}. \end{aligned} \quad (49)$$

For example, setting the phase factor to zero and using the complex dimensions of the class of fractal strings related to $\mathcal{E}^{(\infty)}$, we have the following hierarchy of the imaginary components of the complex dimensions:

$$d_y^{(n)} = \frac{2\pi n}{\ln 2} (\phi)^{j-1}. \quad (50)$$

Following the Fractal Tree of life description of [5] one has the following identification:

$$\epsilon_0 = T_0 - T_c. \quad \epsilon = T - T_c. \quad (51)$$

where T_c is the critical time marking the end of an evolutionary process which began at T_0 . And T is the time variable. One can see that the function L has peaks at discrete values of time $T_n^{(j)}$ that accelerate toward the critical dates according to a log-periodic law:

$$\frac{T_n^{(j)} - T_c}{T_0 - T_c} = 2^{-n(1+\phi)^{j-1}} < 1. \quad (52)$$

where the label j is associated with the particular string or branch \mathcal{L}_j , generated by the scaling ratios $2^{-(1+\phi)^{j-1}}$, and the label n represents the level height within each particular string/branch. The arguments of the cosine function will then be:

$$\left[\frac{2\pi(\phi)^{j-1}}{\ln 2} \right] \ln 2^{n(1+\phi)^{j-1}} = (1+\phi)^{j-1} \phi^{j-1} 2\pi n = 2\pi n. \quad \phi(1+\phi) = 1. \quad (53)$$

and hence, we will have peaks spaced at log-periodic intervals. We can then model the fractal evolutionary tree process of Nottale et al [5] in terms of fractal strings/sprays; i.e. in terms of a sequence of temporal intervals of lengths $T_n^{(j)} - T_c$! associated with a given branch \mathcal{L}_j . Hence, the fractal tree of life envisioned by [5] fits very naturally within the context of fractal strings and their complex dimensions described in the book [1].

If Physics is supposed to explain all of natural phenomena then it must be able to explain Biology and Evolution in Nature. The fractal paradigm described here on the basis of fractal strings/sprays will be a nice starting point. Since we live in four spacetime dimensions it would not be surprising that $d = 4$ will be a relevant scaling exponent.

5. Further Topics: p -adic numbers, Quark Confinement and the Fine Structure Constant

In this last section we will discuss further the p -adic Topology and the p -adic internal symmetries of quasi-crystals and their relation to the Fibonacci modules, the Dirichlet zeta function and the ring of $Z[\tau]$ [10] before embarking into other topics.

Quasi-periodic point sets have been interpreted by many authors as quasi-crystals with p -adic internal symmetries and p -adic Topology [10]. One can generalize the fractal strings/sprays construction of [1] to spaces with p -adic Topology instead of Euclidean one. These will be then called p -adic Fractal Strings and Sprays. Ordinary p -adic strings have been studied extensively in the Literature [11]. The reason why p -adic fractal strings may be very relevant at Planck scales is due to the Non-Archimedean geometry: one cannot naively add/subtract lengths as we ordinary do in ordinary Euclidean geometry, since the Planck scale is taken in Extended Scale Relativity [12] to be the minimum length. One requires to use an ultrametric p -adic distance which reinforces the minimum length principle. In fact we have shown that based on the ideas of C-spaces, Clifford manifolds, one can reproduce the current results of kappa-deformed Poincare theories of gravity [12].

A large number of quasi-periodic tilings (tesselations of the plane,) display these p -adic properties. A class of Z -modules, the Fibonacci modules, with both crystallographic structures and scaling invariance for the ring $Z[\tau]$, the ring of integers in the quadratic field $Q[\tau]$, where $\tau = 1 + \phi$, have been studied by [10]. The number of self-similar submodules was encapsulated by the Dirichlet zeta series associated with the Fibonacci module M :

$$\zeta_M(s) = \sum \frac{a(m)}{m^s} = \sum_{\tilde{M}} \frac{1}{[M : \tilde{M}]^s}. \quad (54)$$

where \tilde{M} runs over the self-similar submodules of M and the quantity defined as $[M : \tilde{M}]$ is the *index* factor, that may or may not bear any relation to Jones theory of subfactors in Knot theory. One of the most

interesting quasi-crystals are those related to 4 dimensions, like the Elser-Sloane quasi-crystals associated with the Hurwitz ring of integral quaternions or the icosian ring [10]. It has been known for some time that Penrose quasi-periodic tiling of the plane is one of the most simplest examples of Noncommutative Geometry whose K -theory groups are in fact related to the ring $Z[\tau]$, connected to the Golden Mean. Using these ideas Selvam-Fadnavis [13] gave an estimate of the inverse fine structure constant:

$$20(1 + \phi)^4 = 100 + 60\phi = 137.082.... \quad (55)$$

based on Penrose quasi-crystal with a five-fold symmetry and a complex growth-process involving the logarithmic spiral with the Golden Mean winding number. There are 5 successive growth-steps in this process, involving a clockwise and counter-clockwise motion giving a net number of $5.2 = 10$. The variance of each step is $2\sigma = 2(1 + \phi)^4$. Hence the total variance is given by eq-(55) and was related to the inverse fine structure constant. El Naschie [14] has obtained the same value as [13] based on a transfinite heterotic string theory formalism, interpreting the number 10 as the dimension of the superstring.

As suggestive as these proposals are, we are more inclined to adopt the bit-string physics and combinatorial hierarchy approaches of the last 30 years by P. Noyes and collaborators [15]. The reason is based once again in the notion of p -adic numbers, p -adic Topology, quasi-crystals, lattices and the Mersenne prime hierarchy. A pure set and lattice construction of *all* the coupling constants in Nature using discrete Clifford algebras was given by T. Smith [16]. A Mersenne prime hierarchy was furnished by Pitkanen in his formulation of Topological Geometrophysics [17]. Based on the ideas of p -adic numbers and transfinite M theory we have shown that the sum of suitable powers of the Golden Mean [18]:

$$1 + (1 + \phi)^2 + (1 + \phi)^{2^2} + (1 + \phi)^{2^3} + (1 + \phi)^3 + (1 + \phi)^{3^2} = \\ 100 + 61\phi = 137.7.... \quad (56)$$

Our proposal was inspired from the p -adic decomposition of 137, for $p = 2$ and has been reinforced lately by the bit-string physics work:

$$137 = 3 + 7 + 127 = (2^2 - 1) + (2^3 - 1) + (2^7 - 1)$$

which admits the 2-adic (binary) expansion:

$$137 = 2^0 + 2^3 + 2^7 = 1 + 8 + 128. \quad (57)$$

Notice that the sum of (3, 7, 127) is indeed 137. The sum of $3 + 7 = 10$ (any connection to the dimension of superstrings?) and that these numbers are nothing but Mersenne primes:

$$M_p = 2^p - 1, \quad p = \text{prime}. \quad (58)$$

A table of well known Mersenne primes is [19]:

$$p = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521.... \quad (59)$$

The Mersenne primes 3, 7, 127 are very special because they belong as well to the table of *doubly* iterated Mersenne primes:

$$M_2 = 2^2 - 1 = 3 \Rightarrow MM_2 = 2^3 - 1 = 7 \Rightarrow M_7 = 2^7 - 1 = 127 \Rightarrow MM_7 = 2^{127} - 1 = \text{prime}. \quad (60)$$

It is unknown if the number:

$$2^{2^{127}-1} - 1$$

is a prime or not. The number $2^{127} - 1$ is a *prime* and is of the order of 10^{38} . These numbers obtained by powers of multiple exponents are called googol-plexus. Bit-string results suggest that these *four* doubly iterated numbers is related to the fact that there are *four* forces in nature. Speculations that the four

division algebras of octonions, quaternions, complex and reals may be connected to the four forces have been invoked by some authors. We don't know if this is true or not, all we can say is that is very suggestive.

The p -adic norm of 137 for $p = 2$ is:

$$||137||_2 = \frac{1}{2^0} = 1. \quad (61)$$

the physical interpretation of this unit 2-adic norm for 137 can be attributed to the bit-string physics result that 1 Coulomb event requires 137 bits [15]:

$$||137||_2 = \frac{1}{2^0} = 1 \Leftrightarrow 1 \text{ Coulomb event} = 137 \text{ bits}. \quad (62)$$

The number $20(1 + \phi)^4 = 100 + 60\phi = 137.082$ [13, 14] does *not* admit a p -adic like expansion like the one given in eq- (56). The interpretation of eq-(56) as sums of powers of the Golden Mean admits also a fractal string/spray interpretation: Each term in the expansion of eq- (56) corresponds precisely to the fractal dimension of a fractal Golden string/spray. Hence, eq- (56) is just:

$$100 + 61\phi = \sum \dim \mathcal{E}^{(n)}. \quad (63)$$

for those sets whose values of n are given by:

$$n = 1, 3, 4, 5, 9, 10$$

respectively. Therefore, the number proposed by us [18] $100 + 61\phi$ is just the sum of the fractal dimensions of a family of fractal sprays constructed by using scaling ratios given by binary powers of the Golden Mean. Our number is irrational and lies between 138 and 137. Let us summarize the bit-string physics results. The Mersenne prime plus the bit-string combinatorial factors give an inverse fine structure value of:

$$\alpha^{-1} = 137[1 - \frac{1}{30 \times 127}]^{-1} = 137.0359674. \quad (64)$$

in astonishing agreement compared with the experimental results: 137.0359895. Recent Astrophysical observations seem to suggest that the fine structure is varying over cosmological times. A variable fine structure constant based on C -spaces has been given by us [12].

Notice that bit-string results involve a *finite* number of concatenation or string bits. The numbers obtained from bit-string physics are *rational* ones, whereas the fractal strings generate *irrational* numbers due to their *transfinite* nature. Irrational Conformal Field Theory has barely been studied in physics. Deep relations to number theory, conformal field theory and the Monster group exist as pointed out by Gannon.

At the next level of the hierarchy we have the *sum* of the next Mersenne prime: $2^{127} - 1$ to the number 137 obtained by the previous sums of the Mersenne primes $3 + 7 + 127 = 137$. Hence this sum allows the bit-string physics evaluation of the Planck mass to proton mass ratio in terms of the Newton constant G_N :

$$(\frac{M_{Planck}}{m_{proton}})^2 = \frac{\hbar c}{G_N m_p^2} = [(2^{127} - 1) + (137)][1 - \frac{1}{3 \times 7 \times 10}] = 1.6933 \times 10^{38}. \quad (65)$$

compared to the experimental value of 1.69358×10^{38} . The Fermi constant; the Electro-weak mixing angle; the proton-electron mass ratio; the pion-electron mass ratio; the muon-electron mass ratio; the pion-nucleon coupling constant; the dark-matter to baryon ratio....have all been calculated to remarkable precision. Is this just numerology? Or there is some deep physical organizing principle in Nature that can be described by these fractal strings? If nature chose the Golden mean to scale its fractal strings why did it do so? The Golden Mean has been known to ancient Greeks, to artists of the Renaissance,Perhaps one must invoke a deeper principle of harmony operating in Nature that selected the Golden Mean over all other numbers. The Golden Mean obeys striking self-similar decompositions involving the unit number 1. Wheeler speculated in the past that information theory may lie at the heart of things. Perhaps Nature's code involves the Golden Mean at its core.

It has been known for some time that there are very strong resemblances between Hadronic Physics and Set theory based on the Banach-Tarski paradox which is one of the most shocking results in Mathematics. It says that a solid ball in R^3 can be broken into 5 pieces that can be re-arranged using rigid motions of 3-space to form *two* balls each of which is the *same* size as the original...These pieces are non-measurable sets and their construction requires the use of the Axiom of Choice.

Augenstein has given convincing arguments [20] that every observed strong interaction involving a hadron-reaction can be envisaged as a paradoxical decomposition or a sequence of paradoxical decompositions. The role of non-Abelian groups in both hadronic physics and this paradoxical decomposition is one mathematical link which connects these two areas. One can envision the nucleon/baryon composed of 3 quarks and mesons composed of 2 quarks giving a total of 5 pieces involved in the Banach-Tarski paradoxical decomposition. Mycielski and Wagon have developed a computer program [21] that allows one to see the essence of this paradox using only triangles in the hyperbolic plane H^2 . They have shown how tilings of the H^2 exist where starting from 3 congruent regions of the H^2 , say Red, Blue, Green (three quarks) one can, by a mere change of the viewpoint of the tiling, to make it clear that each set is congruent also to its complement ! Thus, these sets are simultaneously $1/2$ and $1/3$ of the hyperbolic plane ! One can then envision the three quarks inside the baryon broken up into myriads of different pieces into three distinct regions of Red, Blue and Green. This would represent a hierarchical structure of the quarks themselves. Klein's celebrated tessellation of the H^2 involved using the (hyperbolic) triangle whose angles were $(\pi/2, \pi/3, \pi/7)$. The fact that the primes (2, 3, 7) appear is also very suggestive. Are there other tessellations of the hyperbolic plane involving Mersenne primes?

A hierarchy of scales using Mersenne primes has also been pointed out by Pitkanen [17]. He has a hierarchy given by:

$$L = \sqrt{M_p} L_o. \quad L_o = 137 \times 10^2 \text{ Planck scale.} \quad (66)$$

The Mersenne primes associated with:

$$p = 127, 107, 89, 61, 2. \quad (67)$$

represent the Compton wavelengths of the electron, proton, W boson and the flucton particle [3] and the GUT scale respectively.

To finalize we will mention how Mersenne primes are connected to a perfect number [19]. A perfect number P is that number that can be written as the sum of *all* of its positive divisors. For example:

$$6 = 2 \times 3 = 1 + 2 + 3. \quad 28 = 4 \times 7 = 1 + 2 + 4 + 7 + 14 \dots \quad (68)$$

A perfect number can be written in terms of a Mersenne prime as:

$$P(p) = (2^{p-1})(2^p - 1) = 2^{p-1} M_p. \quad (69)$$

The curious fact is that the perfect numbers $P(p)$ coincides with the number of generators of the groups $SO(2^p)$ given by

$$\frac{1}{2}(2^p)(2^p - 1) = P(p). \quad (70)$$

The most relevant example is the number of generators of $SO(32) = SO(2^5)$ which is the group associated with the anomaly free open superstring in $D = 10$ and has 496 generators:

$$\frac{1}{2} 32 \times 31 = 16 \times 31 = 496 \text{ generators.} \quad 31 = 2^5 - 1 = \text{Mersenne prime.}$$

It is undoubtedly that number theory will bring us more surprises in the future.

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